

Virtual holomorphic sections and Donaldson's construction of symplectic submanifolds

Jean-Paul Mohsen *

November 9, 2016

Donaldson proved (in [2]) that if L is a suitable positive line bundle over a closed symplectic manifold X , then, for k sufficiently large, the tensor power L^k admits sections whose zero sets are symplectic submanifolds of X (the sections are approximately holomorphic and they satisfy some uniform transversality condition). The construction relies on the following observation : the local geometry of the bundles L^k near any point $p \in X$, after a normalization, converges to a model holomorphic Hermitian line bundle K over (some ball in) the tangent space $T_p X$. In this note, we will describe this phenomenon in detail and exploit it to reformulate Donaldson's theorem as a compactness result : near each point p , the sections he obtains accumulate to holomorphic sections of K (that we call "virtual sections" since they are not sections of any bundle over X) and their uniform transversality properties correspond to transversality properties of their limits. Of course, similar considerations apply to all constructions based on Donaldson's techniques (e.g. [1], [4]).

Acknowledgements. I want to thank Emmanuel Giroux for many important suggestions.

1 Virtual sections

Let $X = (X, \omega, J, g)$ be a closed almost-Kähler manifold with a prequantization L (a prequantization is a Hermitian line bundle over X with a unitary connection having curvature $-i2\pi\omega$). The charts we will use are normal coordinates with respect to the metric $g_k = kg$. To be more precise, let $B \subset \mathbb{C}^n$ denote the unit ball, with $n = \frac{1}{2} \dim_{\mathbb{R}} X$. Fix, for every large integer k , a chart $\varphi_k : B \rightarrow X$ satisfying two conditions :

- (1) It is an exponential map for the metric g_k (i.e. given any unit vector $v \in \mathbb{C}^n$, the curve $t \mapsto \varphi_k(tv)$ is a geodesic with g_k -length 1 velocity vector).
- (2) The differential $D\varphi_k(0)$ is a \mathbb{C} -linear map.

Since φ_k is a local diffeomorphism, one can transfer to B the renormalized almost-Kähler structure ($\omega_k = k\omega$, J , $g_k = kg$) and it is well known that this

**jean-paul.mohsen@univ-amu.fr*

almost-Kähler structure tends to the standard flat Kähler structure on B , as $k \rightarrow \infty$, in the \mathcal{C}^∞ -topology.

The following observation is well-known to experts : the local geometry of the bundle L^k converges to the geometry of a model line bundle. Fix some unitary radially flat isomorphism between the pullback line bundle $\varphi_k^* L^k$ and the trivial Hermitian line bundle over B . Use this isomorphism to transfer the connection of $\varphi_k^* L^k$ to the trivial line bundle. Then, as $k \rightarrow \infty$, this unitary connection of the trivial line bundle tends to some model connection ∇^∞ defined by :

$$\nabla^\infty = d - i\pi \sum_{\alpha=1}^n (x_\alpha dy_\alpha - y_\alpha dx_\alpha).$$

There is a more conceptual description : the model connection ∇^∞ is the only radially trivial connection with curvature $-i2\pi \sum_{\alpha=1}^n dx_\alpha \wedge dy_\alpha$.

Warning. Let s be a section of the trivial bundle $B \times \mathbb{C} \rightarrow B$. We say that s is holomorphic if it is holomorphic for the connection ∇^∞ . Although the section s is a function, it is not the usual concept of holomorphic function. For example, the function $\exp(-\frac{\pi}{2} \sum_{\alpha=1}^n |z_\alpha|^2)$ is a holomorphic section and, more generally, the section s of $B \times \mathbb{C} \rightarrow B$ is holomorphic if and only if the function $s \exp(\frac{\pi}{2} \sum_{\alpha=1}^n |z_\alpha|^2)$ is holomorphic in the usual sense.

This set of tools is well-known to experts. We will use it to study sequences of sections. Fix a \mathcal{C}^∞ -smooth section s_k of L^k for every integer $k \geq 1$. The two following definitions play an important role in our reformulation of Donaldson's theory.

Definition 1. We say that the sequence (s_k) is *tame* if it satisfies the following compactness condition.

For every subsequence (k_l) of the sequence of positive integers, for every sequence (φ_l) of charts satisfying conditions (1) and (2) above (for the integer k_l) and for every sequence (j_l) where j_l denotes a unitary radially flat isomorphism between the trivial line bundle $B \times \mathbb{C} \rightarrow B$ and the pullback bundle $\varphi_l^* L^{k_l}$, if σ_l denotes the section of the trivial bundle $B \times \mathbb{C} \rightarrow B$ corresponding to the pullback section $\varphi_l^* s_{k_l}$ via the isomorphism j_l , then the sequence (σ_l) has a subsequence (σ_{l_m}) which converges (on B), in the \mathcal{C}^∞ -topology.

Definition 2. The limit of (σ_{l_m}) is called a *virtual section* of the tame sequence (s_k) .

We emphasize that we *didn't* assume that all charts φ_l have the same center.

Let's state an informal principle. Let (s_k) be a tame sequence. If the sections s_k satisfy some closed condition then one may infer that the virtual sections satisfy some corresponding condition. We won't be more specific about this principle (we won't even explain the meaning of the word closed). We will be limited to an example : if the sections s_k are holomorphic then the virtual sections are holomorphic.

Of course, concerning open conditions, it goes in the opposite direction. For example, if all virtual sections are transverse to 0 then s_k is transverse to 0,

for $k \gg 1$. If, in addition, the zero sets of virtual sections are symplectic submanifolds in B (for the symplectic form $\sum_{\alpha=1}^n dx_{\alpha} \wedge dy_{\alpha}$), then the zero set of s_k is a symplectic submanifold in X , for $k \gg 1$. Note that every complex submanifold is symplectic. Hence one get the following proposition.

Proposition 3. *For every integer $k \geq 1$, let s_k be a C^{∞} -smooth section of L^k . Suppose that (s_k) is a tame sequence and suppose that every virtual section of the sequence (s_k) is holomorphic and transverse to 0. Then, for all $k \gg 1$, the zero set of s_k is a codimension 2 symplectic submanifold in X .*

In the integrable case (that is, X is Kahler), the sections s_k we will consider are often holomorphic whereas in the non-integrable case (X almost-Kahler), typically, the virtual sections are holomorphic but the sections s_k aren't.

2 Transversality theorems

To compare with, let's state a classical algebraic geometry theorem.

Theorem 4. *Suppose J is integrable. Then, for $k \gg 1$, there exists a holomorphic section s_k of L^k which is transverse to 0.*

Proof. Kodaira embedding theorem implies that, for $k \gg 1$, there are no base points. Hence, almost every section is transverse to 0, by Bertini theorem.

Let's first state Donaldson's theorem in the integrable case.

Theorem 5. *Suppose J is integrable. Then, for every $k \geq 1$, one can choose a holomorphic section s_k of L^k such that :*

- (1) *The sequence (s_k) is tame.*
- (2) *The virtual sections of the sequence (s_k) are transverse to 0.*

One may describe this theorem as a variant of theorem 4. The variant has the advantage of being easily transferable to symplectic geometry. Of course this was Donaldson's main goal and most applications of his techniques are symplectic and contact results. It is known that if X is almost-Kahler, then, in general, one can't get holomorphic sections. Nevertheless, we get asymptotic holomorphy.

Theorem 6. *For every $k \geq 1$, one can choose a C^{∞} -smooth section s_k of L^k such that :*

- (1) *The sequence (s_k) is tame.*
- (2) *The virtual sections of the sequence (s_k) are holomorphic and transverse to 0.*

(Hence, for $k \gg 1$, the section s_k is transverse to 0 and, by Proposition 3, the zero set of s_k is a codimension 2 symplectic submanifold.)

Proof (of theorem 5 and theorem 6). Donaldson's techniques (in [2], see

also [3]) produce sections s_k satisfying two families of estimates and some uniform transversality condition. Let's reformulate these conditions : the bounds on s_k and its derivatives imply that the sequence (s_k) is tame, the bound on $\bar{\partial}s_k$ implies that virtual sections are holomorphic and uniform transversality implies that virtual sections are transverse to 0. (Of course, in the integrable case, Donaldson's sections are holomorphic.)

As noted by Donaldson, uniform transversality bounds the Riemannian geometry of the zero set of s_k . For example, one get the following result.

Proposition 7. *Let (s_k) be a tame sequence. Suppose every virtual sequence is transverse to 0. Then the absolute value of the sectional curvature of the zero set of s_k (for every k such that s_k is transverse to 0), if one calculates it with the metric $g_k = kg$, is lower than some bound which doesn't depend on k .*

(Hence, if one prefers to calculate with the metric g , the absolute value of the sectional curvature is lower than some linear function of k .)

Proof. Let u_k denote the supremum of the absolute value of the curvature of the zero set of s_k . If some virtual section is transverse to 0, then the curvature of the zero sets of any corresponding subsequence tends to the curvature of the zero set of the virtual section. Hence every subsequence of the sequence (u_k) admits some convergent subsubsequence.

References

- [1] D. Auroux, *Estimated transversality in symplectic geometry and projective maps*, in "Symplectic Geometry and Mirror Symmetry", Proc. 4th KIAS International Conference, Seoul (2000), World Scientific, 2001, 1-30 (math.SG/0010052).
- [2] S. Donaldson, *Symplectic submanifolds and almost-complex geometry*, J. Diff. Geom. **44** (1996), 666-705.
- [3] S. Donaldson, *Lefschetz pencils on symplectic manifolds*, J. Diff. Geom. **53** (1999), 205-236.
- [4] A. Ibort, D. Martínez-Torres et F. Presas, *On the construction of contact submanifolds with prescribed topology*, J. Diff. Geom **56** (2000), 235-283.